

# COMPLEMENTS OF GRAPHS OF MEROMORPHIC FUNCTIONS AND COMPLETE VECTOR FIELDS

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**ABSTRACT.** Given a meromorphic function  $s : \mathbb{C} \rightarrow \mathbb{P}^1$ , we obtain a family of fiber-preserving dominating holomorphic maps from  $\mathbb{C}^2$  onto  $\mathbb{C}^2 \setminus \text{graph}(s)$  defined in terms of the flows of complete vector fields of type  $\mathbb{C}^*$  and of an entire function  $h : \mathbb{C} \rightarrow \mathbb{C}$  whose graph does not meet  $\text{graph}(s)$ , which was determined by Buzzard and Lu. In particular, we prove that the dominating map constructed by these authors to prove the dominability of  $\mathbb{C}^2 \setminus \text{graph}(s)$  is in the above family. We also study the complement of a double section in  $\mathbb{C} \times \mathbb{P}^1$  in terms of a complex flow. Moreover, when  $s$  has at most one pole, we prove that there are infinitely many complete vector fields tangent to  $\text{graph}(s)$ , describing explicit families of them with all their trajectories proper and of the same type ( $\mathbb{C}$  or  $\mathbb{C}^*$ ), if  $\text{graph}(s)$  does not contain zeros; and families with almost all trajectories non-proper and of type  $\mathbb{C}$ , or of type  $\mathbb{C}^*$ , if  $\text{graph}(s)$  contains zeros. We also study the dominability of  $\mathbb{C}^2 \setminus A$  when  $A \subset \mathbb{C}^2$  is invariant by the flow of a complete holomorphic vector field.

## CONTENTS

1. Introduction	1
2. Complement of the graph of a meromorphic function	4
3. Complements of double sections	7
4. Complete vector fields tangent to graphs	9
5. Dominability and complete vector fields	14
References	18

## 1. INTRODUCTION

A complex manifold  $M$  of dimension two (a surface) is (holomorphically) *dominable* (by  $\mathbb{C}^2$ ) if there is a holomorphic map  $f : \mathbb{C}^2 \rightarrow M$  with Jacobian determinant not identically zero. The map  $f$  is called a *dominating map*. Note that, in general,  $f$  might be non-surjective. For instance,  $M = \mathbb{C}^2$  and  $f(x, y) = (x, xy)$ . As interesting property, one easily obtains that if  $M$  is dominable, there exists a holomorphic map from  $\mathbb{C}$  to  $M$  whose image is not contained in any

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complex subvariety of dimension one of  $M$ , what is called a holomorphic image of  $\mathbb{C}$  in  $M$  which is not analytically degenerated. In particular, this property implies that  $M$  can not be hyperbolic.

The study of surfaces dominable by  $\mathbb{C}^2$  has been developed in the last years by G. Buzzard and S. Lu, see [5] and [6] (references therein for classical results). In many cases, the existence of a holomorphic image of  $\mathbb{C}$  in  $M$  which is not analytically degenerated implies the dominability of  $M$  (see [5, §3 and §4] for  $M$  compact, and [5, §5] for  $M$  non-compact and algebraic). In general, it is not easy to know if the complement of a given analytic curve  $C$  in a non-hyperbolic  $M$  is dominable or not. Even if  $M = \mathbb{C}^2$ ,  $\mathbb{C} \times \mathbb{P}^1$  or  $\mathbb{P}^2$  the question is not entirely solved. On the other hand, one sees that the study of dominability of the complement in  $\mathbb{P}^2$  of a smooth cubic [5, §5.1] (see §2), and in particular, the complement in  $\mathbb{C}^2$  of the graph of a meromorphic function, are behind of the methods to construct explicit dominating maps in the algebraic setting [5] and in other contexts, as the complement in  $\mathbb{C} \times \mathbb{P}^1$  of a double section [6].

In this paper, we use complete vector fields to study dominability problems in the two dimensional case. We will recall some definitions (for more details, see [4], [9], and references therein). Let  $X$  be a holomorphic vector field on  $M$ . Associated to  $X$  there is a differential equation:

$$\varphi'_z(t) = X(\varphi_z(t)), \quad \varphi_z(0) = z \in M,$$

whose local solutions  $\varphi_z$  define the local flow of  $X$  in a neighbourhood of  $(0, z) \in \mathbb{C} \times M$ . Given any point  $z \in M$ , the local solution  $\varphi_z$  can be extended by analytic continuation along paths in  $\mathbb{C}$ , beginning from  $t = 0$ , to a maximal connected Riemann surface  $\pi_z : \Omega_z \rightarrow \mathbb{C}$ , which is a Riemann domain over  $\mathbb{C}$ . The map  $\varphi_z : \Omega_z \rightarrow M$  is said to be a solution and its image  $C_z$  is called the trajectory of  $X$  through  $z$ . We say that a trajectory  $C_z$  of  $X$  is proper if its topological closure  $\overline{C}_z$  in  $M$  defines an analytic curve in  $M$  of pure dimension one.

If  $\Omega_z$  is equal to  $\mathbb{C}$  (as domain in  $\mathbb{C}$ ) for all  $z$  in  $M$ , we say that  $X$  is complete. In this case, each trajectory  $C_z$  is a Riemann surface uniformized by  $\mathbb{C}$ . If  $M$  is Stein (in particular if  $M = \mathbb{C}^2$ ):

- Any trajectory of  $X$  is of type (= analytically isomorphic to)  $\mathbb{C}$  or  $\mathbb{C}^*$ .
- There is a pluripolar set  $E \subset M$  invariant by the flow of  $X$  such that every complex trajectory  $C_z$  with  $z \in M \setminus E$  is of the same type. In particular, we say that  $X$  is of type  $\mathbb{C}$  (resp.  $\mathbb{C}^*$ ) if  $C_z$  is of type  $\mathbb{C}$  (resp.  $\mathbb{C}^*$ ) for a  $z \in M \setminus E$ .
- Any trajectory of type  $\mathbb{C}^*$  is proper. Moreover, if  $X$  is of type  $\mathbb{C}^*$ , there is a meromorphic first integral for  $X$ , and hence all the trajectories of  $X$  are proper.

Let us summarize the results of this article by sections.

• Section 2.

In [5, Theorem 5.2], it is proved that the complement of the graph of a meromorphic function  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  in  $\mathbb{C}^2$  is dominable. In fact, an explicit fiber-preserving

surjective dominating map  $\Phi$  from  $\mathbb{C}^2$  to  $\mathbb{C}^2 \setminus \text{graph}(s)$  is constructed in terms of an entire function  $h : \mathbb{C} \rightarrow \mathbb{C}$  whose graph does not meet  $\text{graph}(s)$  and the entire map  $\Psi(t, w) = (e^{tw} - 1)/t$  (see (II) of §2). One of the main motivations of this article is to see if  $\Phi$  can be given using a complex flow.

We define a family  $\{Z^u\}_{u \in \mathbb{C}(z)}$  of complete vector fields on  $\mathbb{C}^2 \setminus \text{graph}(s)$ , of type  $\mathbb{C}^*$ , whose trajectories are contained in vertical lines (Proposition 1). It allows us to obtain a family  $\{f^u\}_{u \in \mathbb{C}(z)}$  of fiber-preserving surjective dominating maps from  $\mathbb{C}^2$  to  $\mathbb{C}^2 \setminus \text{graph}(s)$ , where each  $f^u$  is defined in terms of the above  $h \in \mathbb{C}(z)$  and of the complex flow of  $Z^u$ . Moreover, by integration of  $Z^u$ , we can explicitly obtain  $f^u$  and show that the map  $\Phi$  constructed by Buzzard and Lu in [5] is one of the dominating maps of  $\{f^u\}_{u \in \mathbb{C}(z)}$  (Theorem 1).

• Section 3.

Let  $D$  be a double section over  $\mathbb{C}$  in  $\mathbb{C} \times \mathbb{P}^1$ . We define a family  $\{W^u\}_{u \in \mathbb{C}(z)}$  of complete vector fields on  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$ , of type  $\mathbb{C}^*$ , whose trajectories are contained in vertical lines (Proposition 2). We obtain a family  $\{g^u\}_{u \in \mathbb{C}(z)}$  of fiber-preserving surjective dominating maps from  $\mathbb{C}^2$  to  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$ , where each  $g^u$  is defined in terms of the complex flow of  $W^u$  and of a holomorphic function  $\sigma : \mathbb{C} \rightarrow \mathbb{P}^1$ , which is determined in [6, Theorem 1.3], whose graph does not meet  $D$  (Theorem 2).

As a corollary, we obtain an alternative proof of [6, Theorem 1.2], using only  $\sigma$  and the completeness of  $W^u$  (Corollary 2).

• Section 4.

Let  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  be a meromorphic function. From Section 2, it follows that each vector field  $Z^u$  on  $\mathbb{C}^2 \setminus \text{graph}(s)$  is complete and never vanishes. One natural question is to ask whether there is a complete holomorphic vector field  $X$  on  $\mathbb{C}^2$  different from  $Z^u$  such that when it is restricted to  $\mathbb{C}^2 \setminus \text{graph}(s)$  is complete and never vanishes. There are two possibilities for such an  $X$  to be analyzed:  $X$  is identically zero on  $\text{graph}(s)$ , or it is not. In this article we will study only the latter one. In particular,  $X$  has at most isolated zeros, and  $X$  is tangent to  $\text{graph}(s)$ .

First, we will prove that  $s$  has at most one pole (Lemma 1).

We study complete vector fields on  $\mathbb{C}^2$  tangent to  $C = \text{graph}(s)$ . We prove that there are infinitely many complete vector fields  $X$ , tangent to  $C$ , and without zeros on  $C$ . In this case, all the trajectories of  $X$  are proper and of type  $\mathbb{C}$ , when  $s$  has no poles ( $C$  of type  $\mathbb{C}$ ), or of type  $\mathbb{C}^*$ , when  $s$  has one pole ( $C$  of type  $\mathbb{C}^*$ ). In the former case, the vector field is analytically equivalent, by a fiber-preserving automorphism of  $\mathbb{C}^2$  that takes  $\text{graph}(s)$  into  $\{t = 0\}$ , to a polynomial vector field. In the latter case, the vector field is analytically equivalent, by a fiber-preserving automorphism of  $\mathbb{C}^2$  that takes  $\text{graph}(s)$  into  $\text{graph}(1/z^k)$ , to a polynomial vector field. For  $s$  without poles, and fixed  $p \in C$ , we prove that there exist infinitely many complete holomorphic vector fields  $X$ , tangent to  $C$  and with only one zero on  $C$  at  $p$ . In this case, moreover,  $X$  can be defined of type  $\mathbb{C}$  and with almost all its trajectories non-proper, or of type  $\mathbb{C}^*$  (§4.1, §4.2, Theorem 4).

• Section 5.

Let  $A$  be a subset of  $\mathbb{C}^2$  invariant by the flow of an holomorphic vector field  $X$ . We study, in some cases, the dominability of  $\mathbb{C}^2 \setminus A$  when  $X|_A$  is complete. If  $A$  is an analytic curve transversal to the foliation defined by  $X$  we determine that  $A$  is the graph of a meromorphic function, after an analytic automorphism, and hence  $\mathbb{C}^2 \setminus A$  is dominable (Proposition 9 and Theorem 5). If  $X$  is a polynomial vector field, with isolated singularities, and  $A = C_z$  is a proper trajectory of type  $\mathbb{C}^*$  where  $\overline{C}_z$  is a singular curve, we prove that there is a dominating holomorphic map  $\Gamma$  from  $\mathbb{C}^2$  to  $\mathbb{C}^2 \setminus C_z$ , such that  $\Gamma(\mathbb{C}^2)$  is biholomorphic to  $(\mathbb{C}^2 \setminus \{xy(y^r - ax^s) = 0\}) \cup \{(0, 0)\}$ , with  $r, s \in \mathbb{N}^+$ ,  $rs \neq 1$  and  $(r, s) = 1$  (Theorem 6).

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## 2. COMPLEMENT OF THE GRAPH OF A MEROMORPHIC FUNCTION

**2.1. Dominability of the complement of a smooth cubic in  $\mathbb{P}^2$ .** Let  $A$  be a smooth cubic in  $\mathbb{P}^2$  and let  $X = \mathbb{P}^2 \setminus A$ . In [5, Proposition 5.1] G. Buzzard and S. Lu proved that  $X$  is holomorphically dominable by  $\mathbb{C}^2$ . The proof is based in two points:

- (I) The existence of a meromorphic function  $s : \mathbb{C} \rightarrow \mathbb{P}^1$ , which is associated to  $A$ , such that dominability of  $\mathbb{C}^2 \setminus \text{graph}(s)$  implies dominability of  $X$ .
- (II) The construction of an explicit fiber-preserving holomorphic map  $\Phi$  from  $\mathbb{C}^2$  onto  $\mathbb{C}^2 \setminus \text{graph}(s)$  with Jacobian determinant not identically zero.

In fact, the proof of (II) implies that the complement in  $\mathbb{C}^2$  of any graph of a meromorphic function  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  must be holomorphically dominable by  $\mathbb{C}^2$  [5, Theorem 5.2].

Let us summarize (II) (see [5, p. 645] for precise details). Note that we can assume that  $s \neq s_\infty$ . In what follows, we will denote by  $\mathbb{C}(z)$  the ring of entire functions of one variable. To prove (II), there are two points:

**(II.1)** *Existence of  $h \in \mathbb{C}(z)$  such that  $\text{graph}(h) \cap \text{graph}(s) = \emptyset$ .*

If  $s = q/q_1$  for  $q, q_1 \in \mathbb{C}(z)$  without common zeros, it is enough to define  $h = s - 1/g$  where  $g = q_1/e^{g_1}$  for  $g_1 \in \mathbb{C}(z)$  such that  $1/g$  and  $s$  have the same principal parts. The existence of  $g_1$  follows from Mittag-Leffler and Weierstrass Theorems as we can see in [5, p. 645].

**(II.2)** *Explicit definition of  $\Phi$  using (II.1) and  $\Psi(t, w) = (e^{tw} - 1)/t$ .*

Note that  $\Psi(t, w)$  is entire on  $\mathbb{C}^2$  because

$$\Psi(t, w) = w + \frac{tw^2}{2!} + \frac{t^2w^3}{3!} + \dots$$

Let us take

$$\phi(z, w) = h(z) - \Psi(g(z), w) = h(z) - \frac{e^{g(z)w} - 1}{g(z)} = s(z) - \frac{e^{g(z)w}}{g(z)}.$$

Note that  $\phi(z, w)$  is entire on  $\mathbb{C}^2$  and equal to  $h(z) - w$  if  $g(z) = 0$ . It holds that

$$\Phi(z, w) = (z, \phi(z, w)) = \left( z, s(z) - \frac{e^{g(z)w}}{g(z)} \right)$$

is a fiber-preserving dominating map from  $\mathbb{C}^2$  onto  $\mathbb{C}^2 \setminus \text{graph}(s)$  with non-vanishing Jacobian determinant.

## 2.2. Buzzard-Lu's results on $\mathbb{C}^2 \setminus \text{graph}(s)$ revisited.

2.2.1. *Family  $\{Z^u\}_{u \in \mathbb{C}(z)}$  of complete vector fields on  $\mathbb{C}^2 \setminus \text{graph}(s)$ .* Let us see in the following proposition that there is a natural family of complete vector fields on  $\mathbb{C}^2 \setminus \text{graph}(s)$ .

**Proposition 1.** *Let  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  be a meromorphic function. There exists a family  $\{Z^u\}_{u \in \mathbb{C}(z)}$  of vector fields on  $\mathbb{C}^2 \setminus \text{graph}(s)$ , which are complete, of type  $\mathbb{C}^*$ , and whose trajectories are contained in vertical lines. Moreover, each  $Z^u$  extends to  $\mathbb{C} \times \mathbb{P}^1$ , as complete vector field of type  $\mathbb{C}^*$ , vanishing only along the graphs of  $s$  and  $s_\infty$ .*

*Proof.* Let us take  $s = q/q_1$  for  $q, q_1 \in \mathbb{C}(z)$  without common zeros. For any  $u \in \mathbb{C}(z)$ , we define

$$Z^u = e^{u(z)} \cdot [q_1(z)w - q_1(z)s(z)] \frac{\partial}{\partial w}.$$

It is easy to check the following facts, which imply the proof of proposition:

- The trajectories of  $Z^u$  are contained in vertical lines.
- $Z^u$  is holomorphic on  $\mathbb{C}^2$  and only vanishes along  $\text{graph}(s)$ .
- $Z^u$  restricted to  $\{z = z_0\}$ , with  $z_0 \in \mathbb{C}$ , is a complete linear vector field.

Then  $Z^u$  is complete on  $\mathbb{C}^2 \setminus \text{graph}(s)$ .

- The trajectory of  $Z^u$  contained in  $\{z = z_0\}$  is either of type  $\mathbb{C}$ , if  $q_1(z_0) = 0$ ; or of type  $\mathbb{C}^*$ , if  $q_1(z_0) \neq 0$ . Therefore,  $Z^u$  is of type  $\mathbb{C}^*$ .

- $Z^u$  extends to  $\mathbb{C} \times \mathbb{P}^1$  as holomorphic vector field (also denoted by  $Z^u$ ) vanishing on  $\{w = \infty\}$ . Thus  $Z^u$  is complete on  $\mathbb{C} \times \mathbb{P}^1$ , and of type  $\mathbb{C}^*$ .  $\square$

Let  $Z^u$  be a vector field of  $\{Z^u\}_{u \in \mathbb{C}(z)}$ . The meromorphic function

$$\mathcal{P}^u(z, w) = \left( \frac{2\pi i}{e^{u(z)} q_1(z)} \right)^2$$

on  $\mathbb{C}^2$  is the period function of  $Z^u$ . Thus for any  $(z, w)$  with  $q_1(z) \neq 0$ , the trajectory of  $Z^u$  through  $(z, w)$ , of type  $\mathbb{C}^*$ , has period  $\sqrt{\mathcal{P}^u}$ . That is,  $\sqrt{\mathcal{P}^u}\mathbb{Z}$  is the discrete subgroup of  $(\mathbb{C}, +)$  defined by the complex times that fix  $(z, w)$  by the flow of  $Z^u$  (details on period function, see [10, page 84]).

Let us see in the following remark that (II.1) can be interpreted in terms of a vector field of  $\{Z^u\}_{u \in \mathbb{C}(z)}$ .

**Remark 1.** Let  $s = q/q_1$  and  $h = s - 1/g$  for  $g, q, q_1 \in \mathbb{C}(z)$  as in (II.1), where  $g = q_1/e^{g_1}$  with  $g_1 \in \mathbb{C}(z)$ .

The vector field  $Z^{-g_1}$  of  $\{Z^u\}_{u \in \mathbb{C}(z)}$ , which is defined by

$$e^{-g_1} \cdot [q_1(z)w - q_1(z)s(z)] \frac{\partial}{\partial w} = [g(z)w - g(z)s(z)] \frac{\partial}{\partial w},$$

has a period function  $\mathcal{P}^{-g_1} = (2\pi i/g)^2$ . It holds that  $s - \sqrt{\mathcal{P}^{-g_1}}/2\pi i = h$ . Note that to determine  $h$  as in (II.1) it is enough to determine  $u \in \mathbb{C}(z)$  such that  $s - \sqrt{\mathcal{P}^u}/2\pi i \in \mathbb{C}(z)$ .

**2.2.2. Family  $\{f^u\}_{u \in \mathbb{C}(z)}$  of fiber-preserving surjective dominating maps.** We will denote by  $\varphi^u$  the global flow of  $Z^u$ . Recall that  $\varphi^u$  is a map from  $\mathbb{C} \times (\mathbb{C} \times \mathbb{P}^1)$  to  $\mathbb{C} \times \mathbb{P}^1$  defined by  $\varphi^u(t, z, w) = (z(t), w(t))$ , where  $(z(t), w(t))$  is the solution of  $Z^u$  through  $z(0) = z$  and  $w(0) = w$ .

Let us see in the following theorem that there is a family of fiber-preserving dominating holomorphic maps from  $\mathbb{C}^2$  onto  $\mathbb{C}^2 \setminus \text{graph}(s)$ . These holomorphic maps will be defined using  $h \in \mathbb{C}(z)$  of (II.1) and the complex flows of the vector fields of  $\{Z^u\}_{u \in \mathbb{C}(z)}$ .

**Theorem 1.** *Let  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  be a meromorphic function. Let us consider  $h \in \mathbb{C}(z)$  as in (II.1) and the complex flow  $\varphi^u$  of  $Z^u$ . Then, the family*

$$\{f^u\}_{u \in \mathbb{C}(z)}$$

*of holomorphic maps*

$$f^u(z, t) = \varphi^u(t, z, h(z)),$$

*is a family of fiber-preserving dominating maps from  $\mathbb{C}^2$  onto  $\mathbb{C}^2 \setminus \text{graph}(s)$ . To be more precise, let  $s = q/q_1$  and  $h = s - 1/g$  for  $g, q, q_1 \in \mathbb{C}(z)$  as in (II.1). Then,  $f^u$  is explicitly given by*

$$f^u(z, t) = \left( z, s(z) - \frac{e^{[e^{u(z)}q_1(z)]t}}{g(z)} \right).$$

*Proof.* By definition, each  $f^u$  is holomorphic. As  $h(z) \neq \infty$  and  $h(z) \neq s(z)$  then  $(z, h(z))$  is not in the set of zeros of  $Z^u$  by Proposition 1. Thus, by completeness of  $Z^u$ , any point  $(z, w) \in \mathbb{C}^2 \setminus \text{graph}(s)$  can be reached by the solution  $(z(t), w(t))$  of  $Z^u$ , with  $z(0) = z$  and  $w(0) = h(z)$ , after time  $t$ , and  $f^u$  is surjective. The vanishing of  $\partial/\partial z$ -component of  $Z^u$  implies that  $f^u$  is fiber-preserving, and  $Z^u(z, h(z)) \neq 0$  implies that the Jacobian determinant of  $f^u$  is not identically zero, since it is  $e^{u(z)}[q_1(z)h(z) - q(z)] \neq 0$  at  $(z, 0)$ .

Let us obtain the explicit expression of  $f^u$ . We take  $(z, h(z))$  in  $\mathbb{C}^2 \setminus \text{graph}(s)$ . Then  $z(t) \equiv z$ . The second component of  $X$  gives the linear differential equation

$$dw/dt = [e^{u(z)}q_1(z)]w(t) - [e^{u(z)}q_1(z)]s(z), \quad w(0) = w = h(z).$$

If  $g(z) \neq 0$ , the above equation can be explicitly solved, and we get

$$\begin{aligned}
w(t) &= \left\{ h(z) - \int_0^t [e^{u(z)} q_1(z)] s(z) e^{-[\int_0^x [e^{u(z)} q_1(z)] ds]} dx \right\} e^{\int_0^t [e^{u(z)} q_1(z)] ds} = \\
&= \left\{ h(z) - [e^{u(z)} q_1(z)] s(z) \int_0^t e^{-[e^{u(z)} q_1(z)] x} dx \right\} e^{[e^{u(z)} q_1(z)] t} = \\
&= [h(z) - s(z)] e^{[e^{u(z)} q_1(z)] t} + s(z) = s(z) - e^{[e^{u(z)} q_1(z)] t} / g(z).
\end{aligned}$$

Note that  $w(t)$  (by definition, it is entire) is equal to  $w - h(z)$  when  $g(z) = 0$ .  $\square$

As a consequence of Theorem 1, let us prove in the following corollary that the dominating map  $\Phi$  constructed by Buzzard and Lu can be defined using  $h \in \mathbb{C}(z)$  of (II.1) and the complex flow  $\varphi^{-g_1}$  of  $Z^{-g_1}$ .

**Corollary 1.** *Given a meromorphic function  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  suppose that  $s = q/q_1$  and  $h = s - 1/g$  for  $g, q, q_1 \in \mathbb{C}(z)$  as in (II.1), where  $g = q_1/e^{g_1}$  with  $g_1 \in \mathbb{C}(z)$ . Then, the dominating map  $f^{-g_1} = \varphi^{-g_1}(t, z, h(z))$  of  $\{f^u\}_{u \in \mathbb{C}(z)}$  is the dominating map  $\Phi(z, t)$  constructed by Buzzard and Lu of (II.2).*

*Proof.* It follows from the explicit expression of  $f^u$  obtained in Theorem 1.  $\square$

### 3. COMPLEMENTS OF DOUBLE SECTIONS

A complex subvariety  $D \subset R \times \mathbb{P}^1$  of dimension one is a double section over a noncompact Riemann surface  $R$  if  $D = \{(z, w) | a(z)w^2 + b(z)w + c(z) = 0\}$ , where  $a, b, c$  are holomorphic functions. Let us assume that  $R = \mathbb{C}$ .

**3.1. Buzzard-Lu's results on  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$  revisited.** G. Buzzard and S. Lu proved in [6, §2] that  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$  is holomorphically dominable. To prove it, they consider three steps:

- (1) The existence of a holomorphic function  $\sigma : \mathbb{C} \rightarrow \mathbb{P}^1$  whose graph does not meet  $D$  [6, Theorem 1.3].
- (2) The existence of a change of coordinates in  $\mathbb{C} \times \mathbb{P}^1$  that transforms  $\sigma$  into  $s_\infty$  so that  $D$  is  $\{(z, w) | w^2 = h(z)\}$  [6, Corollary 1.4].
- (3) The explicit construction of a fiber-preserving dominating map from  $\mathbb{C}^2$  to  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$  [6, Theorem 1.2] using (1) and (2).

**3.1.1. Family  $\{W^u\}_{u \in \mathbb{C}(z)}$  of complete vector fields on  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$ .** Let us see in the following proposition that there is a natural family of complete vector fields on  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$ .

**Proposition 2.** *Let  $D$  be a double section over  $\mathbb{C}$ . There exists a family  $\{W^u\}_{u \in \mathbb{C}(z)}$  of vector fields on  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$ , which are complete, of type  $\mathbb{C}^*$ , and whose trajectories are contained in vertical lines. Moreover, each  $W^u$  vanishes only along  $D$ .*

*Proof.* For any  $u \in \mathbb{C}(z)$ , one defines

$$W^u = e^{u(z)} \cdot [a(z)w^2 + b(z)w + c(z)] \frac{\partial}{\partial w}.$$

It is easy to check the following facts, which imply the proof of the proposition:

- The trajectories of  $W^u$  are contained in vertical lines.
- $W^u$  is holomorphic on  $\mathbb{C} \times \mathbb{P}^1$  and it only vanishes along  $D$ .
- $W^u$  on  $\{z = z_0\} \cup \{\infty\} \simeq \mathbb{P}^1$  is a complete holomorphic vector field, with (one or two) zeros where  $D$  intersects  $\{z = z_0\} \cup \{\infty\}$ . Therefore,  $W^u$  is complete on  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$ .
- The trajectory of  $W^u$  contained in  $\{z = z_0\}$ , with  $z_0 \in \mathbb{C}$ , is of type  $\mathbb{C}^*$ , if  $D$  intersects  $\{z = z_0\}$  at two points or of type  $\mathbb{C}$  if  $D$  intersects  $\{z = z_0\}$  at one point. In particular,  $Z^u$  is of type  $\mathbb{C}^*$ .  $\square$

**3.1.2. Family  $\{g^u\}_{u \in \mathbb{C}(z)}$  of fiber-preserving surjective dominating maps.** One can define analogously as in Theorem 1 a family of fiber-preserving dominating holomorphic maps from  $\mathbb{C}^2$  onto  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$  using (1) and the flow of  $W^u$ . The proof of the following theorem is left to the reader.

**Theorem 2.** *Let  $D \subset \mathbb{C} \times \mathbb{P}^1$  be a double section over  $\mathbb{C}$ . Let us consider the holomorphic map  $\sigma : \mathbb{C} \rightarrow \mathbb{P}^1$  whose graph does not meet  $D$ , according to (1), and the flow  $\psi^u$  of  $W^u$ . Then, the family*

$$\{g^u\}_{u \in \mathbb{C}(z)}$$

*of holomorphic maps*

$$g^u(z, t) = \psi^u(t, z, \sigma(z)),$$

*is a family of fiber-preserving dominating maps from  $\mathbb{C}^2$  onto  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$ .*

As a corollary we obtain an alternative proof of [6, Theorem 1.2].

**Corollary 2.** *Let  $D \subset \mathbb{C} \times \mathbb{P}^1$  be a double section over  $\mathbb{C}$ . Then, the explicit fiber-preserving dominating map from  $\mathbb{C}^2$  to  $(\mathbb{C} \times \mathbb{P}^1) \setminus D$  defined in (3) can be similarly obtained using (only) (1) and the flow of any  $W^u$  of  $\{W^u\}_{u \in \mathbb{C}(z)}$ .*

**3.1.3. Some applications.**

**Proposition 3.** *Let  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  be a meromorphic function. Then,*

- (i) *There exist  $h \in \mathbb{C}(z)$  and a meromorphic function  $\hat{s} : \mathbb{C} \rightarrow \mathbb{P}^1$  such that  $\text{graph}(s) \cap \text{graph}(h)$ ,  $\text{graph}(h) \cap \text{graph}(\hat{s})$  and  $\text{graph}(\hat{s}) \cap \text{graph}(s)$  are empty.*
- (ii) *There exists a complete vector field  $\hat{W}$  on  $\mathbb{C} \times \mathbb{P}^1$ , vanishing only along  $\text{graph}(h) \cup \text{graph}(\hat{s})$ , whose trajectories are all of type  $\mathbb{C}^*$ .*
- (iii) *There exist a double section  $\hat{D} \subset \mathbb{C} \times \mathbb{P}^1$  over  $\mathbb{C}$  and a fiber-preserving holomorphic map from  $\mathbb{C}^2$  onto  $(\mathbb{C} \times \mathbb{P}^1) \setminus \hat{D}$  of the form  $F(z, t) = (z, H(z, t))$ , such that  $F$  on each vertical fiber  $\{z = z_0\}$  is of the form  $e^{c_{z_0}t}$ , with a constant  $c_{z_0} \neq 0$  depending on  $z_0$ , modulo an automorphism of  $\mathbb{P}^1$  mapping the two points in  $\hat{D} \cap \{z = z_0\}$  to 0 and  $\infty$ .*



*Proof.* Let  $s = q/q_1$  and  $h = s - 1/g$  for  $g, q, q_1 \in \mathbb{C}(z)$  as in (II.1). To deduce (i), let  $\hat{s} = \sigma$ , where  $\sigma : \mathbb{C} \rightarrow \mathbb{P}^1$  is the map whose graph does not meet the double section  $D = \{q_1(z)(w - h(z))(w - s(z)) = 0\}$  [6, Theorem 1.3].

To prove (ii), take  $\hat{s} = \hat{q}/\hat{q}_1$  with  $\hat{q}, \hat{q}_1 \in \mathbb{C}(z)$  without common zeros, and define  $\hat{W} = [\hat{q}_1(z)(w - h(z))(w - \hat{s}(z))] \partial / \partial w$ .

To prove (iii), define the double section  $\hat{D} \subset \mathbb{C} \times \mathbb{P}^1$  over  $\mathbb{C}$  given by the set of zeros of  $\hat{W}$ , and  $F(z, t) = \hat{\varphi}(t, z, s(z))$  with  $\hat{\varphi}$  the flow of  $\hat{W}$ .  $\square$

#### 4. COMPLETE VECTOR FIELDS TANGENT TO GRAPHS

Let  $X$  be a holomorphic vector field on  $\mathbb{C}^2$ . We will denote by  $\mathcal{F}(X)$  the holomorphic foliation defined by  $X$ . Note that  $\mathcal{F}(X)$  is a foliation by curves on  $\mathbb{C}^2$  with isolated singularities.

Let  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  be a meromorphic function. From Proposition 1, it follows that each vector field  $Z^u$  on  $\mathbb{C}^2 \setminus \text{graph}(s)$  is complete and never vanishes. Note that  $\mathcal{F}(Z^u)$  is defined by vertical lines and is transversal to  $\text{graph}(s)$ .

One natural question is to ask whether there is a complete holomorphic vector field  $X$  on  $\mathbb{C}^2$  different from  $Z^u$  such that when it is restricted to  $\mathbb{C}^2 \setminus \text{graph}(s)$  is complete and never vanishes. There are two possibilities for such an  $X$  to be analyzed.

- If  $X$  is identically zero on  $\text{graph}(s)$  we can assume that  $\text{graph}(s)$  is invariant by  $\mathcal{F}(X)$ , for otherwise (the proof of) Proposition 9 implies that  $X$  is of the form  $Z^u$  after an analytic automorphism. If moreover  $X$  verifies one of the following properties:

- $X$  defines a quasi-algebraic flow [9],
- $X$  has a meromorphic first integral [9], or
- $X$  is polynomial [3],

one concludes that  $s$  has at most one pole because its set of zeros is an algebraic curve in  $\mathbb{C}^2$  with components of type  $\mathbb{C}$  or  $\mathbb{C}^*$ , and thus  $\text{graph}(s)$  is of type  $\mathbb{C}$  or  $\mathbb{C}^*$ .

On the other hand, all known complete vector fields verify one of the above properties. Nevertheless, in general, it is unknown whether  $s$  can have more than one pole.

**Question.** *Is there a complete holomorphic vector field  $X$  on  $\mathbb{C}^2$ , identically zero on  $\text{graph}(s)$ , such that  $\text{graph}(s)$  is invariant by  $\mathcal{F}(X)$ , when  $s$  has more than one pole?*

- If  $X$  is not identically zero on  $\text{graph}(s)$ ,  $X$  has at most isolated zeros on  $\text{graph}(s)$ . Then,  $X$  is tangent to  $\text{graph}(s)$ , and  $\text{graph}(s)$  is invariant by  $\mathcal{F}(X)$ . In this section, we will work under these assumptions.

Let us see in following lemma that  $s$  has at most one pole and that  $\text{graph}(s)$  is of type  $\mathbb{C}$  or  $\mathbb{C}^*$ .

**Lemma 1.** *Let  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  be a meromorphic function. If there is a complete holomorphic vector field  $X$  on  $\mathbb{C}^2$ , which is tangent to  $\text{graph}(s)$  and with at*

most isolated zeros on  $\text{graph}(s)$ , then  $s$  has at most one pole. Let us consider the trajectory  $C_z$  of  $X$  contained in  $\text{graph}(s)$ , then

- If  $s \in \mathbb{C}(z)$  (no poles),  $C_z$  is of type  $\mathbb{C}$  or  $\mathbb{C}^*$ , being  $C_z$  respectively equal to  $\text{graph}(s)$  or  $\text{graph}(s) \setminus \{p\}$ , where  $p$  is the unique zero of  $X$  over  $\text{graph}(s)$ , or
- If  $s$  has only one pole,  $C_z$  is equal to  $\text{graph}(s)$  and of type  $\mathbb{C}^*$ .

*Proof.* Let  $P \subset \mathbb{C}^2$  be the set of zeros of  $X$ . Then  $C_z = \text{graph}(s) \setminus P$  is of type  $\mathbb{C}$  or  $\mathbb{C}^*$ . As  $X|_{C_z}$  is complete, it extends as a complete holomorphic vector field on  $\text{graph}(s)$  making zeros on  $\text{graph}(s) \cap P$ .

If  $C_z \simeq \mathbb{C}$ , then  $\text{graph}(s) \cap P$  is empty. Otherwise  $\text{graph}(s) \simeq \mathbb{P}^1$ , which is not possible. Then  $C_z = \text{graph}(s)$  and  $s \in \mathbb{C}(z)$ .

If  $C_z \simeq \mathbb{C}^*$ , then  $\text{graph}(s) \cap P$  is empty or has only one point  $p$ . Otherwise  $X$  is not complete on  $\text{graph}(s)$ . In the first case,  $C_z = \text{graph}(s)$  and  $s$  has only one pole. In the second case,  $C_z \cup \{p\} = \text{graph}(s) \simeq \mathbb{C}$  and  $s \in \mathbb{C}(z)$ .  $\square$

**4.1. Vector fields tangent to  $\text{graph}(s)$  if  $s \in \mathbb{C}(z)$ .** Let us construct two families of complete holomorphic vector fields on  $\mathbb{C}^2$  tangent to  $\text{graph}(s)$  when  $s \in \mathbb{C}(z)$ .

**Proposition 4.** *Let  $s : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic map. Then, there are two families of complete vector fields, which are tangent to  $\text{graph}(s)$  and analytically equivalent by a fiber-preserving automorphism that takes  $\text{graph}(s)$  into the line  $\{t = 0\}$  to one of the polynomial vector fields:*

(i)

$$X_1 = (ax + b) \frac{\partial}{\partial x} + A(x)t \frac{\partial}{\partial t}$$

where  $a, b \in \mathbb{C}$ ,  $A \in \mathbb{C}[x]$ ,  $A \not\equiv 0$ , or

(ii)

$$X_1 = at \frac{\partial}{\partial t} + A(x^m t^n) \cdot \left( nx \frac{\partial}{\partial x} - mt \frac{\partial}{\partial t} \right)$$

where  $a \in \mathbb{C}$ ,  $m, n \in \mathbb{N}^+$ ,  $(m, n) = 1$ ,  $A \in \mathbb{C}[y]$ ,  $A \not\equiv 0$ , and  $y = x^m t^n$ .

*Proof.* A polynomial vector field  $X_1$  as (i) or (ii) is complete [3, Theorem] and leaves invariant  $\{t = 0\}$ . On the other hand, if  $\phi$  is the automorphism given by  $\phi(x, t) = (x, t + s(x)) = (z, w)$ , since  $\phi(\{t = 0\}) = \text{graph}(s)$ , it is enough to define the two families as  $\phi_* X_1$ , with  $X_1$  as (i) or (ii).  $\square$

**Proposition 5.** *Let  $s : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic map. Then, there is a family of complete vector fields tangent to  $\text{graph}(s)$ , which have all their trajectories proper and of type  $\mathbb{C}$ .*

*Proof.* It is sufficient to take  $\phi_* X_1$  as in proof of Proposition 4, with  $X_1$  as (i),  $a = 0$  and  $b \neq 0$ .  $\square$

**Proposition 6.** *Let us consider  $s \in \mathbb{C}(z)$  and  $p \in \text{graph}(s)$ . Then, there is a family of complete holomorphic vector fields tangent to  $\text{graph}(s)$  and vanishing only at  $p$ . Moreover, infinitely many of them are of type  $\mathbb{C}$  and with almost all*

their trajectories non-proper, and infinitely many other of them are of type  $\mathbb{C}^*$ . Therefore, there are infinitely many vector fields of this family not equivalent by an analytic automorphism.

*Proof.* Let us take  $X = \phi_* X_1$  as in proof of Proposition 4, with  $X_1$  as (i), where  $a \neq 0$ ,  $b = 0$  and  $A(0) \neq 0$ ; or with  $X_1$  as (ii), where  $a \neq 0$ ,  $A(0) \neq 0$  and  $a - A(0)m \neq 0$ . Let  $\lambda$  be the quotient of the eigenvalues of the linear part of  $DX(z, w)$  at the unique zero of  $X$ , that we can assume as  $p$ . Note that in the former case  $\lambda = a/A(0)$ , and in the latter case  $\lambda = A(0)n/(a - A(0)m)$ . Let us take  $a$  and  $A(0)$  so that  $\lambda \notin \mathbb{R}^- \cup \{0\}$ . According to Poincaré's Theorem, the foliation  $\mathcal{F}(X)$  is given by  $x\partial/\partial x + \lambda y\partial/\partial y$  in suitable coordinates  $(x, y)$  around  $p$  [2, p. 10]. In particular, in a neighborhood  $U_p \subset \mathbb{C}^2$  of  $p$ , if  $C_z$  is a trajectory with  $z \in U_p$ ,  $C_z \cap U_p$  contains a level set of  $x^{-\lambda}y$ .

- If  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , for any  $z \in U_p$ ,  $C_z \cap U_p$  accumulates  $\{xy = 0\}$  [7, p. 120]. Hence,  $C_z$  is not proper and of type  $\mathbb{C}$ , and  $X$  is of type  $\mathbb{C}$  (see §1).
- If  $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ , for any  $z \in U_p$ ,  $C_z \cap U_p$  contains a real subvariety of dimension three [7, p. 120]. Hence,  $C_z$  is not proper and of type  $\mathbb{C}$ , and  $X$  is of type  $\mathbb{C}$  (see §1).
- If  $\lambda = p/q \in \mathbb{Q}^+$ , for any  $z \in U_p$ ,  $C_z \cap U_p$  is a punctured disk. Hence,  $C_z$  is proper and of type  $\mathbb{C}^*$ , and  $X$  is of type  $\mathbb{C}^*$  (see §1).

Finally, if  $\beta(z, w) = (u, v)$  is an analytic automorphism of  $\mathbb{C}^2$ , since the quotient of the eigenvalues of the linear part  $DY(u, v)$  of  $Y = \beta_* X$  at  $p' = \beta(p)$  is  $\lambda$ , the last sentence of the statement follows.  $\square$

**4.2. Vector fields tangent to  $\text{graph}(s)$  if  $s$  has one pole.** Let us construct two families of complete holomorphic vector fields on  $\mathbb{C}^2$  tangent to  $\text{graph}(s)$  when  $s$  has one pole.

**Theorem 3.** *Let  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  be a meromorphic function with only one pole of order  $k \in \mathbb{N}^+$ . Then, there are two families of complete vector fields of type  $\mathbb{C}^*$ , which are tangent to  $\text{graph}(s)$ , and analytically equivalent by a fiber-preserving automorphism that takes  $\text{graph}(s)$  into  $\text{graph}(1/z^k)$  to one of the polynomial vector fields:*

(iii)

$$Y_0 = az \frac{\partial}{\partial z} + \left( A(z)w + \frac{-(ak + A(z))}{z^k} \right) \frac{\partial}{\partial w},$$

with

$$A(z) = -ak + A_0(z), \quad A_0 \in z^k \cdot \mathbb{C}[z], \quad \text{or}$$

(iv)

$$Y_0 = a \left( \frac{wz^k - 1}{z^k} \right) \frac{\partial}{\partial w} + A(z^{m-nk} (wz^k - 1)^n) \cdot \left[ nz \frac{\partial}{\partial z} + \left( -mw + \frac{(m-nk)}{z^k} \right) \frac{\partial}{\partial w} \right].$$

with  $m > nk$  and

$$A(y) = a/(m - nk) + A_0(y), \quad A_0 \in y^k \cdot \mathbb{C}[y], \quad y = z^{m-nk}(wz^k - 1)^n.$$

**Remark 2.** If  $Y_0$  is as (iii) or (iv) of Theorem 3, according to [9, Théorème 4] (see also §5.1), there is an analytic automorphism  $\phi$  of  $\mathbb{C}^2$  such that

$$\phi_* Y_0 = \lambda(t) \cdot \left( z \frac{\partial}{\partial z} - kw \frac{\partial}{\partial w} \right), \quad \lambda \in \mathbb{C}(t), \quad t = wz^k.$$

As a consequence of Theorem 3, there are two complete holomorphic vector fields of type  $\mathbb{C}^*$  tangent to  $\text{graph}(s)$ , generically transversal, and such that their corresponding foliations are analytically but not algebraically equivalent.

**Proof of Theorem 3.** We will assume that  $z = 0$  is the pole of  $s$ . Then  $s(z) = s_0(z)/z^k$ , with  $s_0 \in \mathbb{C}(z)$ ,  $s_0(0) \neq 0$  and  $k \in \mathbb{N}^+$ . Let us see in the following lemma that (II.1) allows us to define an automorphism that takes  $\text{graph}(s)$  into  $\text{graph}(1/z^k)$ .

**Lemma 2.** *Under the conditions stated above, there exists a fiber-preserving automorphism  $\varphi$  of  $\mathbb{C}^2$  such that*

$$\varphi(\text{graph}(s)) = \text{graph}(1/z^k).$$

*Proof.* Let us define  $\varphi(z, w) = (z, e^{-g_1(z)}[w - h(z)]) = (x, y)$ , where  $h = s - 1/g$  for  $g = z^k e^{-g_1} \in \mathbb{C}(z)$  by (II.1). Note that  $\varphi$  is an automorphism and that  $\varphi^{-1}(x, y) = (x, e^{g_1(x)}y + h(x)) = (z, w)$ . By definition,  $\varphi(z, s(z)) = (z, 1/z^k)$ .  $\square$

Let us see in the following proposition an application of Lemma 2.

**Proposition 7.** *Let  $Z$  be a holomorphic vector field on  $\mathbb{C}^2$ . Let us suppose that  $Z$  is tangent to the curve  $\{wz^k - s_0(z) = 0\}$ , with  $s_0 \in \mathbb{C}(z)$ ,  $s_0(0) \neq 0$  and  $k \in \mathbb{N}^+$ . Then,  $Z$  is analytically equivalent, by a fiber-preserving analytic automorphism  $\varphi$  of  $\mathbb{C}^2$ , to a vector field  $Y$  tangent to  $\{wz^k - 1 = 0\}$ .*

*Proof.* It is enough to define  $Y = \varphi_* Z$ , with  $\varphi$  as in Lemma 2.  $\square$

Let us see in the following proposition that the vector fields of Theorem 3 can be defined using  $\varphi$  of Lemma 2.

**Proposition 8.** *Let  $\varphi$  be the automorphism of Lemma 2. Then, the vector field  $X = \varphi^* Y_0$ , where  $Y_0$  is as (iii) or (iv) of Theorem 3, is complete on  $\mathbb{C}^2$ , of type  $\mathbb{C}^*$ , and tangent to  $\text{graph}(s)$ .*

*Proof.* Let  $X_1$  be as (i), with  $b = 0$ , or as (ii) of Proposition 4. Then  $X_1$  leaves invariant  $\{t = 0\}$  and  $\{x = 0\}$ . If  $\alpha$  is the map given by  $\alpha(x, t) = (x, t + 1/x^k) = (z, w)$ , one has that  $\alpha$  is a biholomorphism of  $\mathbb{C}^* \times \mathbb{C}$  such that

$\alpha(\{t = 0\}) = \text{graph}(1/z^k)$ . Then  $\alpha_*X_1$  is a complete holomorphic vector field on  $\mathbb{C}^* \times \mathbb{C}$  tangent to  $\text{graph}(1/z^k)$ . On the other hand,

$$\alpha_*X_1 = \begin{pmatrix} 1 & 0 \\ \frac{-k}{z^{k+1}} & 1 \end{pmatrix} \cdot X_1(z, w - 1/z^k).$$

An explicit computation shows that  $\alpha_*X_1$  is holomorphic on  $\mathbb{C}^2$ , and then complete on  $\mathbb{C}^2$ , if  $\alpha_*X_1 = Y_0$ , with  $Y_0$  as in the statement. On the other hand, since the quotient of the eigenvalues of the linear part of  $DY_0(p)$  at the unique zero  $p$  of  $Y_0$  is  $-1/k$ ,  $Y_0$  is of type  $\mathbb{C}^*$ , and then  $X$  is as well. Finally, from Lemma 2, it follows that  $X$  must be complete and tangent to  $\text{graph}(s)$  since  $\varphi^{-1}(\text{graph}(1/z^k)) = \text{graph}(s)$ .  $\square$

Then, we have finished the proof of Theorem 3.

**4.3. Summary.** The main results of §4.1 and §4.2 may be summarized in the following theorem:

**Theorem 4.** *Let  $C$  be an analytic curve defined by the graph of a meromorphic function  $s : \mathbb{C} \rightarrow \mathbb{P}^1$  with at most one pole. Then,*

- *If  $s$  has no poles,*
  - *There exist infinitely many complete holomorphic vector fields tangent to  $C$ , without zeros on  $C$ , and such that all their trajectories are proper and of type  $\mathbb{C}$ . Moreover, if  $X$  is one of them, then  $C$  is a trajectory  $C_z$  of  $X$  and  $X$  is analytically equivalent, by a fiber-preserving analytic automorphism of  $\mathbb{C}^2$  that takes  $C_z$  into a line, to a polynomial vector field.*
  - *For any  $p \in C$ , there exist infinitely many complete holomorphic vector fields tangent to  $C$ , with only one zero on  $C$ , which is  $p$ , and not equivalent by an analytic automorphism. If  $X$  is one of them,  $C \setminus \{p\}$  is a trajectory  $C_z$  of  $X$  of type  $\mathbb{C}^*$ . Moreover,  $X$  can be defined of type  $\mathbb{C}$  and with almost all its trajectories non-proper, or of type  $\mathbb{C}^*$ . In particular, there are infinitely many of these vector fields not equivalent by an analytic automorphism.*
- *If  $s$  has one pole,*
  - *There exist infinitely many complete holomorphic vector fields tangent to  $C$ , without zeros on  $C$ , and such that all their trajectories are proper and of type  $\mathbb{C}^*$ . Moreover, if  $X$  is one of them, then  $C$  is a trajectory  $C_z$  of  $X$  and  $X$  is analytically equivalent, by a fiber-preserving analytic automorphism of  $\mathbb{C}^2$  that takes  $C$  into  $\{wz^k - 1 = 0\}$ , to a polynomial vector field.*

**Remark 3.** Under the assumption of this section, that is,  $X$  is not identically zero on  $\text{graph}(s)$ , the fact that  $\text{graph}(s)$  must be of type  $\mathbb{C}$  or  $\mathbb{C}^*$  by Lemma 1 is the obstruction to define complete vector fields tangent to  $\text{graph}(s)$  when  $s$  has more than one pole.

The construction of an automorphism using (II.1), as  $\varphi$  in Lemma 2, that takes  $\text{graph}(s)$  into  $\text{graph}(1/q_1)$  also works when  $s$  has more than one pole. However, in this case our procedure only produces vector fields of the form  $Z^u$ .

The reason is that  $\alpha(x, t) = (x, s(x) + t)$  is a biholomorphism of  $\mathbb{C}^2$  minus at least two vertical lines (see §4.2), and then the only complete polynomial vector field that we can push forward by  $\alpha$  is the vertical one (Picard Theorem).

On the other hand, the existence of a complete holomorphic vector field  $X$  identically zero on  $\text{graph}(s)$ , and such that  $\text{graph}(s)$  is invariant by  $\mathcal{F}(X)$ , when  $s$  has more than one pole, would imply that there are other complete vector fields until now unknown, as we have mentioned at the beginning of this section.

## 5. DOMINABILITY AND COMPLETE VECTOR FIELDS

Let  $A$  be a subset of  $\mathbb{C}^2$  invariant by the flow of a holomorphic vector field  $X$ . In this section we will study, in some cases, the dominability of  $\mathbb{C}^2 \setminus A$  when  $X|_A$  is complete.

### 5.1. $\mathbb{C}^2 \setminus C$ for $C$ transversal to $\mathcal{F}(X)$ and $X$ complete on $\mathbb{C}^2 \setminus C$ .

**Proposition 9.** *Let  $C$  be an analytic curve in  $\mathbb{C}^2$ . If  $C$  is not invariant by  $\mathcal{F}(X)$  and  $X$  is complete on  $\mathbb{C}^2 \setminus C$  then:*

- (i)  *$X$  vanishes on  $C$ , and then  $X$  is complete and of type  $\mathbb{C}^*$ ,*
- (ii) *Up to an analytic automorphism of  $\mathbb{C}^2$ ,  $C$  is the graph of a meromorphic function  $s : \mathbb{C} \rightarrow \mathbb{P}^1$ .*

*Proof.* To obtain (i), assume that  $X|_C \neq 0$  and derive a contradiction. Take  $p \in C$  such that  $X(p) \neq 0$ . Suppose that the trajectory  $C_z$  of  $X$  through  $p$  is transversal to  $C$  at  $p$ . Locally, in a neighborhood  $V$  of  $p$ , taking coordinates  $(z, w)$  with  $p = (0, 0)$ ,  $X|_V = \partial/\partial z$ . On the other side,  $X|_{C_z \setminus C}$  complete implies that  $X|_{(C_z \setminus C) \cap V}$  has a zero at 0, which is not possible. We conclude that  $X|_C \equiv 0$  and  $X|_V = h\partial/\partial z$ , for  $h$  holomorphic on  $V$ , vanishing on  $C \cap V$ , which we assume  $\{z = 0\}$ , at order 1 by completeness. Therefore  $X$  has infinitely many trajectories of type  $\mathbb{C}^*$  whose topological boundary in  $\mathbb{C}^2$  is one point in  $C$ . In particular,  $X$  is complete and of type  $\mathbb{C}^*$ , and defining a proper flow [9].

To prove (ii), the structure of  $X$  is well known [9, Théorème 4]. Up to an analytic change of coordinates,  $X$  is one of the following vector fields:

(1)

$$a(z) \frac{\partial}{\partial w},$$

with  $a \in \mathbb{C}(z)$ .

(2)

$$[g(z)(w - s(z))] \frac{\partial}{\partial w},$$

with  $g \in \mathbb{C}(z)$ , and  $s$  meromorphic such that  $gs \in \mathbb{C}(z)$ .

(3)

$$\lambda(t) \cdot \left( nz \frac{\partial}{\partial z} + mw \frac{\partial}{\partial w} \right),$$

with  $\lambda(t) \equiv \lambda \in \mathbb{C}^*$ ,  $m, n \in \mathbb{N}^*$ ,  $(m, n) = 1$  or  $\lambda \in \mathbb{C}(t)$ ,  $t = z^{-m}w^n$ ,  $-m, n \in \mathbb{N}^*$ ,  $(m, n) = 1$ .

(4)

$$\frac{\gamma(t)}{z^\ell} \cdot \left( n z^{\ell+1} \frac{\partial}{\partial z} - [(m + n\ell)z^\ell w + mp(z) + nzp(z)] \frac{\partial}{\partial w} \right),$$

with  $m, n \in \mathbb{N}^*$ ,  $(m, n) = 1$ ,  $\ell \in \mathbb{N}$ ,  $p \in \mathbb{C}[z]$  of degree  $< \ell$  with  $p(0) \neq 0$  if  $\ell > 0$  or  $p \equiv 0$  if  $\ell = 0$ ,  $\gamma \in \mathbb{C}(t)$  vanishing at  $t = 0$  at order  $\geq \ell/m$ ,  $t = z^m(z^\ell w + p(w))^n$ .

Cases (1), (3) and (4) are not possible because their set of zeros are invariant by  $\mathcal{F}(X)$ . Therefore,  $X$  is as (2) and  $C$  is equal to  $\text{graph}(s)$ .  $\square$

**Remark 4.** Note that (ii) implies that  $C$  is biholomorphic to  $\mathbb{C}$  minus the set of poles of a meromorphic function in  $\mathbb{C}$ .

**Theorem 5.** *Let  $C$  be an analytic curve in  $\mathbb{C}^2$ . If  $C$  is not invariant by  $\mathcal{F}$  and  $X$  is complete on  $\mathbb{C}^2 \setminus C$  then  $\mathbb{C}^2 \setminus C$  is holomorphically dominable.*

*Proof.* It follows from Theorem 1 and Proposition 9.  $\square$

### 5.2. $\mathbb{C}^2 \setminus C_z$ for $\overline{C}_z$ singular and $X$ polynomial and complete on $C_z$ .

Let us study the dominability of the complementary in  $\mathbb{C}^2$  of a trajectory  $C_z$  of a vector field  $X$ . Here, we only treat the case of a polynomial vector field  $X$  with isolated zeros that is complete on a proper trajectory  $C_z$ . Recall that  $C_z$  is proper if the topological closure  $\overline{C}_z$  in  $\mathbb{C}^2$  is an analytic curve. In this situation,  $C_z$  is of type  $\mathbb{C}$  or  $\mathbb{C}^*$ . The trajectory  $C_z$  is algebraic if  $\overline{C}_z$  is an algebraic curve.

5.2.1.  $C_z$  of type  $\mathbb{C}$ . Note that  $\overline{C}_z = C_z$ . If  $C_z$  is algebraic,  $C_z = \{y = 0\}$ , after a polynomial automorphism by Abhyankar-Moh-Suzuki theorem [8]. If  $C_z$  is nonalgebraic,  $C_z$  defines a leaf of an algebraic foliation  $\mathcal{F}(X)$  with all its ends (one) planar, isolated and properly embedded in  $\mathbb{C}^2$ . Then  $C_z = \{y = 0\}$  after an analytic automorphism [1]. Then,  $\mathbb{C}^2 \setminus C_z$  is biholomorphic to  $\mathbb{C} \times \mathbb{C}^*$  (complement of graph of  $s = 0$ ), and  $\mathbb{C}^2 \setminus C_z$  is dominable.

5.2.2.  $C_z$  of type  $\mathbb{C}^*$ . Note that  $\overline{C}_z = C_z$  or  $\overline{C}_z = C_z \cup \{p\}$  with  $X(p) = 0$ . Let us study the case where  $\overline{C}_z$  is a singular curve. Necessarily,  $\overline{C}_z$  has only one singularity, say  $p$ , and  $\overline{C}_z = C_z \cup \{p\}$  with  $X(p) = 0$ .

**Theorem 6.** *Let  $C_z$  be a proper trajectory of type  $\mathbb{C}^*$  of a polynomial vector field  $X$  on  $\mathbb{C}^2$ . If  $\overline{C}_z$  is singular and  $X|_{C_z}$  is complete, then  $\overline{C}_z = \{y^r - ax^s = 0\}$ , with  $a \neq 0$ ,  $r, s \in \mathbb{N}^+$ ,  $r \cdot s \neq 1$  and  $(r, s) = 1$ , after an analytic automorphism. In particular,*

- (i) *There is a non-surjective holomorphic dominating map  $\Gamma$  from  $\mathbb{C}^2$  to  $\mathbb{C}^2 \setminus C_z$ , such that*
- (ii) *Its image  $\Gamma(\mathbb{C}^2)$  is biholomorphic to*

$$(\mathbb{C}^2 \setminus \{xy(y^r - ax^s) = 0\}) \cup \{(0, 0)\}.$$

**Proof of Theorem 6.** If  $C_z$  is algebraic,  $\overline{C}_z$  is defined by  $\{y^r - ax^s = 0\}$  after a polynomial automorphism, by Lin-Zaidenberg's theorem [11].

If  $C_z$  is nonalgebraic, let us see in the following proposition that  $\overline{C}_z$  is also given by above equation after an analytic automorphism.

**Proposition 10.** *Let  $C_z$  be a proper trajectory of type  $\mathbb{C}^*$  of  $X$  such that  $\overline{C}_z$  is singular and  $X|_{C_z}$  is complete. If  $C_z$  is not algebraic, then  $X$  is one of the following polynomial vector fields, up to a polynomial automorphism of  $\mathbb{C}^2$ :*

(1)

$$X = \lambda x \frac{\partial}{\partial x} + [a(x)y + c(x)] \frac{\partial}{\partial y},$$

with  $\lambda/a(0) \in \mathbb{Q}^+ \setminus \{\mathbb{N}^+ \cup 1/\mathbb{N}^+\}$ .

(2)

$$X = x[n f(x^m y^n) + \alpha] \frac{\partial}{\partial x} - y[m f(x^m y^n) + \beta] \frac{\partial}{\partial y},$$

with  $m, n \in \mathbb{N}^*$ ,  $f(z) \in z \cdot \mathbb{C}[z]$ ,  $\alpha, \beta \in \mathbb{C}$  such that  $\alpha m - \beta n \in \mathbb{C}^*$  and  $-\alpha/\beta \in \mathbb{Q}^+ \setminus \{\mathbb{N}^+ \cup 1/\mathbb{N}^+\}$ .

*Proof.* The fact that  $\overline{C}_z = C_z \cup \{p\}$  implies that  $X$  is one of the following vector fields, up to a polynomial automorphism [4, p. 663]:

(a)

$$\lambda x \frac{\partial}{\partial x} + [a(x)y + b(x)] \frac{\partial}{\partial y},$$

where  $a, b \in \mathbb{C}[x]$ , and  $a(0), \lambda \in \mathbb{C}^*$ .

(b)

$$x[n f(x^m y^n) + \alpha] \frac{\partial}{\partial x} - y[m f(x^m y^n) + \beta] \frac{\partial}{\partial y},$$

with  $m, n \in \mathbb{N}^*$ ,  $f(z) \in z \cdot \mathbb{C}[z]$ ,  $\alpha, \beta \in \mathbb{C}$  such that  $\beta/\alpha \in \mathbb{Q}^-$  and  $\alpha m - \beta n \in \mathbb{C}^*$ .

(c)

$$x[n S + \alpha] \frac{\partial}{\partial x} + \left\{ -\frac{[nT + m(x^\ell y + p(x))] S + \alpha T}{x^\ell} \right\} \frac{\partial}{\partial y},$$

for  $m, n, \ell \in \mathbb{N}^*$ ,  $\alpha \in \mathbb{C}^*$ ,  $p(x) \in \mathbb{C}[x]$  of degree  $< \ell$ ,  $p(0) \neq 0$ ,  $T = \ell x^\ell y + x p'(x)$ ,  $S = f(x^m (x^\ell y + p(x))^n)$  with  $f(z) \in z \cdot \mathbb{C}[z]$ , and where

$$[n x p'(x) + m p(x)] S + \alpha x p'(x) \in x^\ell \cdot \mathbb{C}[x, y].$$

Let us analyze such an  $X$  when  $\overline{C}_z$  is a singular curve.

• *Case (a).* Let us use well-known results about singularities of vector fields around  $p = (0, 0)$  [2, pp. 11-16]. Let  $\lambda_1 = \lambda$  and  $\lambda_2 = a(0)$  be the eigenvalues of the linear part  $DX(p)$  of  $X$  at  $p$ .

If  $\lambda_1/\lambda_2 \notin \mathbb{Q}^+$ , there are only two separatrices of  $\mathcal{F}$  through  $p$ , which are smooth and transversal at  $p$ , which is impossible. If  $\lambda_1/\lambda_2 = r/s \in \mathbb{Q}^+$ , there are two possibilities:

If  $r/s \notin \mathbb{N}^+ \cup 1/\mathbb{N}^+$ , according to Poincaré's linearization theorem,  $X$  is  $rz\partial/\partial z + sw\partial/\partial w$  in certain coordinates around  $p$ . Then  $z^s/w^r$  is a local first integral, and this possibility can occur.



If  $r/s \in \mathbb{N}^+ \cup 1/\mathbb{N}^+$ , according to Poincaré-Dulac's normal form theorem,  $X$  is  $z\partial/\partial z + (nw + \epsilon z^n)\partial/\partial w$ , with  $\epsilon \in \{0, 1\}$ ,  $n = r/s$  or  $s/r \in \mathbb{N}^+$  around  $p$ . If  $\epsilon = 0$ ,  $z^n/w$  is a local first integral. Then all the separatrices through  $p$  are smooth, which is not possible. If  $\epsilon = 1$ ,  $ze^{-w/z^n}$  is a local first integral. Thus there are no separatrices different from  $\{x = 0\}$ , which is also impossible. Therefore,  $\lambda/a(0) \in \mathbb{Q}^+ \setminus \{\mathbb{N}^+ \cup 1/\mathbb{N}^+\}$ .

• *Case (b).* The eigenvalues of  $DX(p)$  are  $\lambda_1 = \alpha$  and  $\lambda_2 = -\beta$ . Then  $\lambda_1/\lambda_2 \in \mathbb{Q}^+$ . One analyzes as in case (a) that  $\lambda_1/\lambda_2 \in \mathbb{Q}^+ \setminus \{\mathbb{N}^+ \cup 1/\mathbb{N}^+\}$ .

• *Case (c).* According to [4, p. 649], if  $H$  is the regular covering map from  $u \neq 0$  to  $x \neq 0$ ,  $(u, v) \mapsto (x, y) = H(u, v) = (u^n, u^{-(m+n\ell)}[v - u^m p(u^n)])$ ,

$$H^*X = [f(v^n) + \alpha/n]u \frac{\partial}{\partial u} + [\alpha m/n]v \frac{\partial}{\partial v}.$$

Then  $X$  has only one zero  $p$ , which is on  $\{x = 0\}$  (invariant by  $X$ ). Working with the expression of  $X$  one obtains that  $DX(p)$  has eigenvalues  $\lambda_1 = \alpha$  and  $\lambda_2 = -\alpha\ell$ . Then  $\lambda_1/\lambda_2 = -1/\ell \notin \mathbb{Q}^+$ , and there are only two separatrices through  $p$ , which are smooth and transversal at  $p$ , which is not possible. Hence (c) does not occur.  $\square$

Let us see in the following proposition the analytic version of Proposition 10.

**Proposition 11.** *Let  $C_z$  be a proper trajectory of type  $\mathbb{C}^*$  of  $X$  such that  $\overline{C}_z$  is singular and  $X|_{C_z}$  is complete. If  $C_z$  is not algebraic,*

$$X = rx \frac{\partial}{\partial x} + sy \frac{\partial}{\partial y},$$

*with  $r, s \in \mathbb{N}^+$ ,  $r \cdot s \neq 1$  and  $(r, s) = 1$ , up to an analytic automorphism of  $\mathbb{C}^2$ .*

*Proof.* By Proposition 10,  $X$  is complete, of type  $\mathbb{C}^*$ , and with only one zero that is moreover the topological boundary in  $\mathbb{C}^2$  of any trajectory of  $X$ . The proof follows from [9, p. 530].  $\square$

According to Proposition 11, modulo an analytic automorphism of  $\mathbb{C}^2$ ,  $C_z$  is contained in a level set of  $y^r/x^s$ , which is a first integral of  $X$ . Therefore,  $\overline{C}_z = \{y^r - ax^s = 0\}$ , being  $p = (0, 0)$  and  $a \neq 0$ . The condition  $(r, s) = 1$  allows us to assume  $pr - qs = 1$  for  $p, q \in \mathbb{N}^+$ . Let us consider  $x = v^q u^r$  and  $y = v^p u^s$  with  $u, v \in \mathbb{C}$ . It holds

$$y^r - ax^s = v^{qs+1}u^{sr} - av^{qs}v^{rs} = u^{rs}v^{qs}(v - a).$$

Hence, it is enough to take a surjective dominating map from  $\mathbb{C}^2$  to  $\mathbb{C}^2 \setminus \{v = a\}$ , and compose it with  $\gamma : \mathbb{C}^2 \setminus \{v = a\} \rightarrow \mathbb{C}^2 \setminus C_z$  defined as  $(u, v) \mapsto \gamma(u, v) = (v^q u^r, v^p u^s)$ , to obtain  $\Gamma$ .

Therefore, we have finished the proof of Theorem 6.

**Remark 5.** Note that for a proper trajectory  $C_z$  of type  $\mathbb{C}^*$  such that  $\overline{C}_z = C_z \cup \{p\}$  with  $X(p) = 0$ ,  $\mathbb{C}^2 \setminus C_z$  is not a manifold. Theorem 6 implies that if  $\overline{C}_z$  is singular, and moreover  $X$  is polynomial with  $X|_{C_z}$  complete,  $\mathbb{C}^2 \setminus C_z$  is a holomorphically dominable set by  $\mathbb{C}^2$ .

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